ARTICLE VII.

New Formulæ relative to Comets. *By E. Nulty, Philadelphia.* *Read September* 21, 1838.

The investigation which I here propose to make, respects the component velocities of a comet, observed in three positions, at consecutive and moderately small intervals of time. It has for its basis the theorem of Maclaurin, as adapted to proximate states of a variable function, and the known expressions for the sun's attractive force on the comet and the earth, referred as usual to rectangular solar axes. The means which I employ are therefore the same as those presented by Lagrange in the Mécanique Analytique, and which Mr Pontécoulant has recently adopted in his Théorie Analytique du Système du Monde, where formulæ for determining the distances and orbits of comets are given with appropriate developments. But the object which I have here in view, is not the same as that of Lagrange, in his celebrated work above mentioned; and my investigation and results are different from those of Mr Pontécoulant, and embrace a wider extent of subject. Similar diversity and extension, in mathematical research, are in perpetual requisition. They constitute an essential and important part of analytic science; and with their peculiar attractions, always lead to useful views and advantageous contrast. As to the instance now adduced, the presumed novelty, and the great accuracy and simplicity of the formulæ which I have obtained, entitle them, I should

hope, to attention and preference. Their mode of investigation I also judge important, and as peculiarly eligible. It has enabled me to exhibit the formulæ hitherto given, as particular states of those just noticed; and besides others equally simple, it has furnished two new and general sets of expressions for the exceptive cases in which the observed latitudes and longitudes of the comet would render the general formulæ doubtful or indeterminate. To the analysis of the principal of these results, and with regard to practical applications, I have adjoined the data of the comet of 1805, and for which I am indebted to the excellent treatise of Mr Pontécoulant. The corresponding velocities I have computed by the formulæ now given, and by others connected with the method of La Place. Their comparison has led me to some remarks with which I conclude this paper, and which I have inserted from an opinion of their analytical and practical importance.

Before I enter on the proposed investigation, I think it may not be improper to observe, that within a few days, I have been favoured with the perusal of Mr Encke's Astronomical, Annual Register (*Astronomische Yahrbuch*) for 1833, in which its distinguished author has given a full and neat analysis of Dr Olbers' method of determining the orbits of comets. The greater part of that analysis I had in fact the earlier pleasure of reading in Dr Bowditch's Appendix to the Third Volume of his Translation of the Mécanique Céleste. But I had not been previously apprized of Mr Encke's remarks on methods which differ from that of Dr Olbers; and in this paper, I would be understood as having no wish to aim at lessening the predilection with which I am now acquainted, and which may be well and reasonably founded. My own mathematical partialities, I am not unwilling to avow. I entertain them on methods and processes of computation which furnish symmetrical and direct results; and in no slight degree am I favourable to that method which is connected with the formulæ now occupying my attention, and which I have endeavoured to present in such form as to merit the approval of Mr Encke, and the author of the Théorie Analytique du Système du Monde.

Consider a comet at any point C in its orbit, and let its place at the distance r from the centre S of the sun, be determined by the rec-

tangular co-ordinates x, y, z. Refer the earth's centre E, to the axes of x, y, supposed to be in the plane of the ecliptic, and let (X, Y) determine its place at the distance R from S. The position of the comet relatively to E, will then depend on the values of $x - \mathrm{X}$, $y - \mathrm{Y}$, z; and if we denote its co-ordinates measured from E by $\rho\alpha$, $\rho\beta$, $\rho\gamma$, we shall have in its position C

$$x = \mathrm{X} + \rho\alpha, \quad y = \mathrm{Y} + \rho\beta, \quad z = \rho\gamma \tag{1}$$

Accent the different letters in these expressions, in order that they may correspond to two different positions of the comet at C′, C″; the first being supposed to precede, the second to follow C, at the comparatively small intervals of time t', t''. The co-ordinates C′, C″, in the direction of the axes of x, and at the end of these intervals, will then be,

$$x' = \mathrm{X}' + \rho'\alpha'; \quad x'' = \mathrm{X}'' + \rho''\alpha''; \tag{2}$$

and corresponding expressions will result in the directions of the other axes of y and z.

The determination of these co-ordinates in terms relative to the intermediate position C of the comet, and to the corresponding place E of the earth, may be effected by M'Laurin's theorem, and the known differentials

$$x_{,,} = -\frac{x}{r^3}, \quad \mathrm{X}_{,,} = -\frac{\mathrm{X}}{\mathrm{R}^3};$$

which express the sun's attractive force on the comet and earth. By means of that theorem, we have the expressions

$$x' = x - x_{,}t' + \tfrac{1}{2}x_{,,}t'^2 - \tfrac{1}{6}x_{,,,}t'^3, \&c., \quad \mathrm{X}' = \mathrm{X} - \mathrm{X}_{,}t' + \tfrac{1}{2}\mathrm{X}_{,,}t'^2 - \tfrac{1}{6}\mathrm{X}_{,,,}t'^3, \&c.$$

which, in virtue of the preceding differentials, take the usual form

$$x' = u'x - v'x_{,}, \quad \mathrm{X}' = \mathrm{U}'\mathrm{X} - \mathrm{V}'\mathrm{X}_{,} \tag{3}$$

the assumed coefficients of x, $x_{,}$, and of X, $\mathrm{X}_{,}$, having in terms of the interval t', the following values:

$$\left.\begin{array}{l} u'=1-\tfrac{1}{2}\,\dfrac{t'^2}{r^3}+\tfrac{1}{2}\,\dfrac{r_{\prime}t'^3}{r^5},\ \&c.;\ v'=t'-\tfrac{1}{6}\,\dfrac{t'^3}{r^3}+\tfrac{1}{4}\,\dfrac{r_{\prime}t'^4}{r^5},\ \&c. \\ U'=1-\tfrac{1}{2}\,\dfrac{t'^2}{R^3}+\tfrac{1}{2}\,\dfrac{R_{\prime}t'^3}{R^5},\ \&c.;\ V'=t'-\tfrac{1}{6}\,\dfrac{t'^3}{R^3}+\tfrac{1}{4}\,\dfrac{R_{\prime}t'^4}{R^3},\ \&c. \end{array}\right\}\quad(4)$$

The forms for x'', X'' in (2) are similar to these, observing to change the single into double accents, as respects u, U; v, V; and also t' into $-t''$; and like expressions will evidently apply in the directions of the axes of y, z.

Substitute (3) and the similar forms for x'', X'' in (2), and then eliminate x by the first of equations (1). There will result the two expressions:

$$v'(x_{\prime}-X_{\prime})=u'\rho\alpha-\rho'\alpha'+v'\xi',\quad v''(x_{\prime}-X_{\prime})=\rho''\alpha''-u''\rho\alpha+v''\xi'',$$

in which, for brevity, we have assumed;

$$v'\xi'=(V'-v')X_{\prime}-(U'-u')X,\ v''\xi''=(V''-v'')X_{\prime}+(U''-u'')X\quad(5)$$

and which, being respectively multiplied by $\dfrac{v''}{v'}$, $\dfrac{v'}{v''}$ and added, will give

$$\left.\begin{array}{l} (x_{\prime}-X_{\prime})(v'+v'')=\left(\dfrac{u'}{v'}\,v''-\dfrac{u''}{v''}\,v'\right)\rho\alpha-\dfrac{v''}{v'}\,\rho'\alpha'+\dfrac{v'}{v''}\,\rho''\alpha''+v''\xi'+v'\xi'' \\ (y_{\prime}-Y_{\prime})(v'+v'')=\left(\dfrac{u'}{v'}\,v''-\dfrac{u''}{v''}\,v'\right)\rho\alpha-\dfrac{v''}{v'}\,\rho'\beta'+\dfrac{v'}{v''}\,\rho''\beta''+v''\eta'+v'\eta'' \\ \qquad z_{\prime}(v'+v'')=\left(\dfrac{u'}{v'}\,v''-\dfrac{u''}{v''}\,v'\right)\rho\alpha-\dfrac{v''}{v'}\,\rho'\gamma'+\dfrac{v'}{v''}\,\rho''\gamma''; \end{array}\right\}\ (6)$$

the two last being formed by analogy from the first.

Expressions apparently more simple than these, might have been found from the preceding values; but the present forms are more convenient for the determination of the component velocities $x_{\prime}, y_{\prime}, z_{\prime}$, which will become known by means of the earth's velocities $X_{\prime}$, $Y_{\prime}$, when we have expressed the geocentric distances ρ, ρ', ρ'', in terms given by observation of the comet in the corresponding positions C, C′, C″.

For this purpose, equate the two values of $x_, - X_,$, which result from the expressions preceding (5); and from these and analogy we obtain

$$\left.\begin{aligned}\left(\frac{u'}{v'}+\frac{u''}{v''}\right)\rho\alpha-\frac{1}{v'}\rho'\alpha'-\frac{1}{v''}\rho''\alpha''&=\xi,\\ \left(\frac{u'}{v'}+\frac{u''}{v''}\right)\rho\beta-\frac{1}{v'}\rho'\beta'-\frac{1}{v''}\rho''\beta''&=\eta,\\ \left(\frac{u'}{v'}+\frac{u''}{v''}\right)\rho\gamma-\frac{1}{v'}\rho'\gamma'-\frac{1}{v''}\rho''\gamma''&=0;\end{aligned}\right\} \quad (7)$$

in which we have put $\xi = \xi'' - \xi'$, $\eta = \eta'' - \eta'$.

Resolve these equations relatively to ρ, ρ', ρ'', with their coefficients; and in order to exhibit the results in brief terms, let the coefficients which are found to affect ξ, η be denoted by

$$\left.\begin{aligned}A &= \beta'\gamma''-\beta''\gamma', \quad A' = \beta''\gamma-\beta\gamma'', \quad A'' = \beta\gamma'-\beta'\gamma;\\ B &= \alpha''\gamma'-\alpha'\gamma'', \quad B' = \alpha\gamma''-\alpha''\gamma, \quad B'' = \alpha'\gamma-\alpha\gamma'.\end{aligned}\right\} \quad (8)$$

These are known from observation of this comet at C, C′, C″. They give the conditional equations

$$\left.\begin{aligned}A\beta + A'\beta' + A''\beta'' = 0, \quad A\gamma + A'\gamma' + A''\gamma'' = 0,\\ B\alpha + B'\alpha' + B''\alpha'' = 0, \quad B\gamma + B'\gamma' + B''\gamma'' = 0;\end{aligned}\right\} \quad (9)$$

and enable us to express the values now under consideration thus:

$$\left(\frac{u'}{v'}+\frac{u''}{v''}\right)\rho = \frac{A\xi+B\eta}{D}, \; -\frac{1}{v'}\rho' = \frac{A'\xi+B'\eta}{D}, \; -\frac{1}{v''}\rho'' = \frac{A''\xi+B''\eta}{D}; \quad (10)$$

the common denominator being either of the forms

$$D = A\alpha + A'\alpha' + A''\alpha'' = B\beta + B'\beta' + B''\beta''; \quad (11)$$

which and (9) will be of immediate use in the simplification of formulæ (6).

Values analogous to these, though very different in their form and object, are made the basis of Lagrange's method of determining the orbits of Comets. They may be seen in his Mécanique Analytique.

In consequence of the supposed proximity of the extreme positions C′, C″, to the mean position C of the comet, it is easily conceived that the denominator D of (10) is a small quantity, liable to be affected by unavoidable errors of observations. We shall therefore eliminate it, and employ, instead of (10), the following values:

$$\left.\begin{aligned} \rho' &= -\rho v'\left(\frac{u'}{v'}+\frac{u''}{v''}\right)\cdot\frac{N'}{N}, \\ \rho'' &= -\rho v''\left(\frac{u'}{v'}+\frac{u''}{v''}\right)\cdot\frac{N''}{N}; \end{aligned}\right\} \qquad (12)$$

in which we have assumed

$$N = A\xi + B\eta,\quad N' = A'\xi + B'\eta,\quad N'' = A''\xi + B''\eta. \qquad (13)$$

These quantities are the numerators of (10); they give, in virtue of the conditional equations (9) and (11)

$$N\alpha + N'\alpha' + N''\alpha'' = D\xi,\quad N\beta + N'\beta' + N''\beta'' = D\eta. \qquad (14)$$

Let now (12) be substituted in the general expressions (6), and let us attend to the preceding forms for $D\xi$, $D\eta$. We shall then obtain the following values:

$$\left.\begin{aligned} x_{,} &= X_{,} + \frac{\rho}{N}\left(\frac{u''}{v''}N'\alpha' - \frac{u'}{v'}N''\alpha'' + wD\xi\right) + \frac{v'\xi'' + v''\xi'}{v'+v''}, \\ y_{,} &= Y_{,} + \frac{\rho}{N}\left(\frac{u''}{v''}N'\beta' - \frac{u'}{v'}N''\beta'' + wD\eta\right) + \frac{v'\eta'' + v''\eta'}{v'+v''}, \\ z_{,} &= \frac{\rho}{N}\left(\frac{u''}{v''}N'\gamma' - \frac{u'}{v'}N''\gamma''\right); \end{aligned}\right\} \qquad (15)$$

in which we have put $w\,(v'+v'') = \frac{u'}{v'}\,v'' - \frac{u''}{v''}\,v$.

These formulæ are well adapted to the proposed determination of the comet's component velocities. The generality which we have intentionally observed in presenting them, has imposed on their last terms a complex appearance. But this will immediately vanish, when they are limited to the particular state in which approximate results are the objects of computation.

For the purpose here intimated, let us employ the expressions (4) and their similar forms with double accents. If the intervals t', t'' be regarded as small quantities, so that the terms which involve the differentials $r_{,}$, $R_{,}$ of the distances r, R, may be rejected, we shall have for the coefficients of N′, N″ in (15), the values

$$\frac{u'}{v'} = \frac{1}{t'}\left(1 - \frac{t'^2}{3r^3}\right), \quad \frac{u''}{v''} = \frac{1}{t''}\left(1 - \frac{t''^2}{3r^3}\right);$$

in which we shall assume for brevity

$$\mu' = 1 - \frac{t'^2}{3r^3}, \quad \mu'' = 1 - \frac{t''^2}{3r^3}; \qquad (16)$$

and to a near approximation r may be supposed equal to unity.

The factor D in the fourth terms of the components $x_{,}$, $y_{,}$, we have already noticed as being a small quantity; and we shall presently see that ξ, η depend on factors which never exceed the radius R. We may therefore to great accuracy take $w = \frac{t'-t''}{t't''}\left(1 - \frac{t't''}{6r^3}\right)$, the second factor of which, analogously to (16), we shall denote by μ.

Substitute the expressions (4) just employed in the assumed quantities (5); and put the factor $\frac{1}{r^3} - \frac{1}{R^3} = k$. We shall obtain, in consequence of the minuteness of k and the intervals t', t'', the two values

$$\xi' = \tfrac{1}{2}k\left(\tfrac{1}{3}t'^2 X_{,} - t'X\right), \quad \xi'' = \tfrac{1}{2}k\left(\tfrac{1}{3}t''^2 X_{,} + t''X\right);$$

whence there results $\xi'' - \xi'$, or

$$\xi = \tfrac{1}{2}k(t' + t'')\left(X - \frac{t'-t''}{3}X_{,}\right);$$

and by substitution, the last term of the component velocity $x_,$ becomes

$$\frac{v'\xi''+v''\xi'}{v'+v''}=\tfrac{1}{6}kt't''\mathrm{X}_,.$$

Values similar to these will evidently result for η and the last term of $y_,$; and if, for the sake of brevity, we put

$$\xi_,=\mathrm{X}-\tfrac{1}{3}(t'-t'')\mathrm{X}_,\,,\quad \eta_,=\mathrm{Y}-\tfrac{1}{3}(t'-t'')\mathrm{Y}_,; \qquad (17)$$

so that the quantities ξ, η may take the form

$$\xi=\tfrac{1}{2}k(t'+t'')\,\xi_,\,,\quad \eta=\tfrac{1}{2}k(t'+t'')\,\eta_,; \qquad (18)$$

we shall immediately perceive that after substitution in formulæ (15), the factors of $\xi_,$, $\eta_,$ will disappear; and that all the terms but the last of $x_,$, $y_,$, will be independent of the small quantity $\frac{1}{2}k$. To eliminate $\frac{1}{2}k$ from these terms, we must have recourse to the first of the equations (10), which in virtue of the values (4) becomes

$$\frac{t'+t''}{t'\,t''}\left(1-\frac{t'\,t''}{3r^3}\right)=\frac{\mathrm{A}\xi+\mathrm{B}\eta}{\mathrm{D}}.$$

From this and the preceding values of ξ, η we get $\frac{1}{2}kt't''=\left(1-\frac{t'\,t''}{3r^3}\right)\cdot\frac{\mathrm{D}\rho}{\mathrm{A}\xi_,+\mathrm{B}\eta_,}$; so that if for conciseness we put the factor $1-\frac{t'\,t''}{3r^3}=\nu$, and

$$\mathrm{M}=\mathrm{A}\xi_,+\mathrm{B}\eta_,\,,\quad \mathrm{M}'=\mathrm{A}'\xi_,+\mathrm{B}'\eta_,\,,\quad \mathrm{M}''=\mathrm{A}''\xi_,+\mathrm{B}''\eta_,\,,$$

the general expressions (15) will take the form

$$\left.\begin{aligned}
x_,&=\mathrm{X}_,+\frac{\rho}{\mathrm{M}}\left\{\frac{\mu''}{t''}\mathrm{M}'\alpha'-\frac{\mu'}{t'}\mathrm{M}''\alpha''-\mu\mathrm{D}\xi_,\left(\frac{t'-t''}{t'\,t''}\right)+\tfrac{1}{3}\nu\mathrm{D}\mathrm{X}_,\right\},\\
y_,&=\mathrm{Y}_,+\frac{\rho}{\mathrm{M}}\left\{\frac{\mu''}{t''}\mathrm{M}'\beta'-\frac{\mu'}{t'}\mathrm{M}''\beta''-\mu\mathrm{D}\eta_,\left(\frac{t'-t''}{t'\,t''}\right)+\tfrac{1}{3}\nu\mathrm{D}\mathrm{Y}_,\right\},\\
z_,&=\frac{\rho}{\mathrm{M}}\left\{\frac{\mu''}{t''}\mathrm{M}'\gamma'-\frac{\mu'}{t'}\mathrm{M}''\gamma''\right\};
\end{aligned}\right\}(19)$$

the several terms of which are completely determined as respects the intervals t', t''. It remains therefore only to assign values to the earth's velocities $X_{,}$, $Y_{,}$; and to the equations (1).

Denote by L the longitude of the earth when the comet is observed in its mean position C. Let ϖ be the longitude of the perihelion of its elliptic orbit, and $e = \sin \varepsilon$ the eccentricity corresponding to the mean distance 1. The value of ϖ in 1801 was 99° 30′ 5″, annual in. = + 1′ 2″. At the same epoch, e was ·01685301, sec. var. = —·000041809. The co-ordinates of the earth in terms of the radius vector R and longitude L, are

$$X = R \cos L, \quad Y = R \sin L;$$

and the known expressions for R and the elementary area described in the instant dt, are

$$R = \frac{\cos \varepsilon^2}{1 + e \sin (L - \varpi)}, \quad R^2 \frac{dL}{dt} = \cos \varepsilon.$$

If the three first of these expressions be differentiated relatively to the time t of which R and L are functions; and $\frac{dL}{dt}$ be eliminated by means of the last expression; there will result

$$\left.\begin{aligned} X_{,} &= p \cos L - q \sin L, \\ Y_{,} &= p \sin L + q \cos L; \end{aligned}\right\}$$

in which we have assumed for brevity

$$p = \tan \varepsilon \sin (L - \varpi), \quad q = \frac{\cos \varepsilon}{R}; \qquad (20)$$

and which, in conjunction with the values of X, Y above given, will make known the component velocities and position of the earth.

The forms which we have here adopted, enable us to express the values of $\xi_{,}$, $\eta_{,}$ in simple terms. We may assume for the coefficients of cos L, sin L in these values;

$$s\cos a = \mathrm{R} - \tfrac{1}{3}(t' - t'')\,p, \quad s\sin a = \tfrac{1}{3}(t' - t'')\,q;$$

from which there will result

$$\tan a = \frac{(t' - t'')q}{3\mathrm{R} - (t' - t'')p}, \quad s = \frac{(t' - t'')q}{3\sin a}; \qquad (21)$$

and we shall have instead of (17), the values

$$\xi_{,} = s\cos(\mathrm{L} - a), \quad \eta_{,} = s\sin(\mathrm{L} - a);$$

in which a is evidently a small arc and s nearly equal to R.

A further simplification of the expressions last given, may be effected without diminishing their accuracy, or rendering more complex the velocities $\mathrm{X}_{,}$, $\mathrm{Y}_{,}$. Conceive the axis of x, the position of which is arbitrary, to be directed so as to form an angle equal to a with the radius vector R. From this position which we suppose to be less advanced than R, the angle $\mathrm{L} - a$ will then take its origin, and we shall have

$$\xi_{,} = s, \quad \eta_{,} = 0; \quad \mathrm{X} = \mathrm{R}\cos a, \quad \mathrm{Y} = \mathrm{R}\sin a; \qquad (22)$$

the corresponding velocities being

$$\left.\begin{aligned} \mathrm{X}_{,} &= p\cos a - q\sin a, \\ \mathrm{Y}_{,} &= p\sin a + q\cos a, \end{aligned}\right\} \qquad (23)$$

in which the first term of $\mathrm{Y}_{,}$ is extremely small.

With respect to the angular quantities in the equations (1), now to be considered; let l denote the geocentric longitude of the comet in its position C; and λ the corresponding geocentric latitude. If we also represent by ρ the curtate distance of the comet's projected place in the plane of the ecliptic, we shall have, in the present position of the axis of x, the following values:

$$\alpha = \cos(a + l - \mathrm{L}), \quad \beta = \sin(a + l - \mathrm{L}), \quad \gamma = \tan\lambda, \qquad (24)$$

and the co-ordinates (1), by adding their squares, will give the equation:

$$r^2 = \mathrm{R}^2 + 2\mathrm{R}\rho \cos(l - \mathrm{L}) + \rho^2 \sec \lambda^2; \qquad (25)$$

which establishes a relation between the curtate distance ρ and the comet's distance r from the centre of the sun.

Expressions similar to these evidently apply to the extreme positions C′, C″ of the comet; and in (19), (25) we may, if requisite, write $\rho \cos \lambda$ instead of ρ, which will then denote the ray drawn from the centre E of the earth to the comet at C.

When the distance r is given, we shall know ρ by means of (25); and in such case all the quantities in (19) being determined, we may compute the values of the required velocities $x_{,}$, $y_{,}$, $z_{,}$. But should r be also unknown as well as ρ; and this is generally the case, the equation (25) will not alone be sufficient, and we must join to it the differential expression

$$x_{,}^2 + y_{,}^2 + z_{,}^2 = \frac{2}{r}, \qquad (26)$$

which we adapt as a known form for the square of the comet's velocity in a parabolic orbit; and which will enable us to obtain the distances r, ρ, and lastly the component velocities $x_{,}$, $y_{,}$, $z_{,}$.

We may now present the expressions (19) in their final terms for computation. They become, in terms of the simplified values of $\xi_{,}$, $\eta_{,}$, of the following form:

$$\left.\begin{aligned} x_{,} &= \mathrm{X}_{,} + \frac{\rho}{\mathrm{A}}\left(\frac{\mu''}{t''}\mathrm{A}'\alpha' - \frac{\mu'}{t'}\mathrm{A}''\alpha'' - \mu\mathrm{D}\frac{t'-t''}{t't''} + \nu\frac{\mathrm{DX}_{,}}{3s}\right), \\ y_{,} &= \mathrm{Y}_{,} + \frac{\rho}{\mathrm{A}}\left(\frac{\mu''}{t''}\mathrm{A}'\beta' - \frac{\mu'}{t'}\mathrm{A}''\beta'' + \nu\frac{\mathrm{DY}_{,}}{3s}\right), \\ z_{,} &= \frac{\rho}{\mathrm{A}}\left(\frac{\mu''}{t''}\mathrm{A}'\gamma' - \frac{\mu'}{t'}\mathrm{A}''\gamma''\right); \end{aligned}\right\} \qquad (\mathrm{A})$$

in which the values of A, A′, A″ from (8) are

$$\mathrm{A} = \beta'\gamma'' - \beta''\gamma', \quad \mathrm{A}' = \beta''\gamma - \beta\gamma'', \quad \mathrm{A}'' = \beta\gamma' - \beta'\gamma;$$

and the corresponding value of D is by (11),

$$\mathrm{D} = \mathrm{A}\alpha + \mathrm{A}'\alpha' + \mathrm{A}''\alpha''.$$

There is but one remark to be made relatively to the numerical conversion of these and the auxiliary quantities μ, μ', &c. and a, s. The intervals t', t'' and also $t'-t''$, given in days, must be expressed in terms of mean solar time, and in parts of radius, by multiplying them by the diurnal factor ·01720213, which corresponds to the solar arc 59′ 8″·2, and of which the logarithm is 8·2355821.

The preceding formulæ (A) are we believe the most convenient and the most accurate that have appeared for the determination of the component velocities of a comet, observed in three positions at comparatively small intervals. The factors μ, μ', &c. have here been first noticed; they give every requisite degree of precision to the values of $x_{,}$, $y_{,}$, $z_{,}$. The computation we have reduced to uniformity and facility by means of the three arcs $a+l-\mathrm{L}$, $a+l'-\mathrm{L}$, $a+l''-\mathrm{L}$, of which the several angular quantities α, α', &c. are functions; and which take place of nine arcs hitherto employed. The particular form given to D is also advantageous. It directly leads to the value of this factor, by the three previously determined quantities A, A′, A″.

When the observations can be taken so that the intervals t', t'' may be equal, the factors μ', μ'', ν will each be expressed by $1-\frac{t'^2}{3r^3}$. The term which depends on $t'-t''$ will then vanish, and we shall have the more simple expressions:

$$\left.\begin{aligned} x_{,} &= \mathrm{X}_{,} + \frac{\rho\mu'}{\mathrm{A}t'}\left(\mathrm{A}'\alpha' - \mathrm{A}''\alpha'' + \frac{\mathrm{DX}_{,}}{3\mathrm{R}}\right), \\ y_{,} &= \mathrm{Y}_{,} + \frac{\rho\mu'}{\mathrm{A}t'}\left(\mathrm{A}'\beta' - \mathrm{A}''\beta'' + \frac{\mathrm{DY}_{,}}{3\mathrm{R}}\right), \\ z_{,} &= \frac{\rho\mu'}{\mathrm{A}t'}\left(\mathrm{A}'\gamma' - \mathrm{A}''\gamma''\right); \end{aligned}\right\} \qquad (\mathrm{A}')$$

in which the quantities A, A′, A″ and D have the same forms as before; but instead of (24), we have

$$\alpha = \cos(l-\mathrm{L}), \quad \beta = \sin(l-\mathrm{L}), \quad \gamma = \tan\lambda,$$

and similar values for α', α'', &c.

In the formulæ hitherto given for this particular case, the factor μ' has not been introduced or noticed; and on this account, and the last terms involving the earth's velocities $X_{,}$, $Y_{,}$, they are less exact than the preceding expressions (A$'$), in which the term $\frac{DY_{,}}{3R}$ at least should be computed as no negligible, minute quantity.

The general formulæ, of which we have now completed the investigation, and also (A$'$), fail in giving accurate results, when the quantities A, A', A'' and D are minute; and they become indeterminate, when these quantities vanish. If we conceive the apparent path of the comet during the interval $t'+t''$ to be a great circle, passing through a point in the ecliptic of which the longitude is less than that of the sun by the small arc a, the tangent of the inclination of the orbit will be evidently expressed by either of the three relations $\frac{\gamma}{\beta}, \frac{\gamma'}{\beta'}, \frac{\gamma''}{\beta''}$; and we shall have then $A=0$, $A'=0$, $A''=0$, and also $D=0$, to which the formulæ (A) and (A$'$) are inapplicable. The case in which these quantities become small, is now obvious without further reflection; and should it be inconvenient or impossible to take new observations of the comet at other places in its orbit, we must abandon formulæ (A), and have recourse to different expressions, capable of determining the component velocities $x_{,}$, $y_{,}$, $z_{,}$.

Such expressions are the following: they have not to our knowledge been hitherto presented.

$$\left.\begin{aligned} x_{,} &= X_{,} + \frac{\rho}{E}\left(\frac{\mu''}{t''}E'\alpha' - \frac{\mu'}{t'}E''\alpha''\right) + K\left(\frac{\varepsilon''\alpha'}{E} - \frac{t'}{\theta} + \frac{X_{,}t't''}{3R\theta}\right) \\ y_{,} &= Y_{,} + \frac{\rho}{E}\left(\frac{\mu''}{t''}E'\beta' - \frac{\mu'}{t'}E''\beta''\right) + K\left(\frac{\varepsilon''\beta'}{E} + \frac{Y_{,}t't''}{3R\theta}\right) \\ z_{,} &= \frac{\rho}{E}\left(\frac{\mu''}{t''}E'\gamma' - \frac{\mu'}{t'}E''\gamma''\right) + K\frac{\varepsilon''\gamma'}{E}. \end{aligned}\right\}(E)$$

There are in fact two sets of formulæ, comprehended in these expressions. We may take in terms of the geocentric longitudes;

$$E = \sin(l'' - l'), \quad E' = \sin(l - l''), \quad E'' \sin(l' - l), \quad \varepsilon'' = \beta'';$$

or when these longitudes vary from each other less than the corresponding latitudes, we may more suitably adopt the values

$$E = \alpha'\gamma'' - \alpha''\gamma', \quad E' = \alpha''\gamma - \alpha\gamma'', \quad E'' = \alpha\gamma' - \alpha\gamma'; \quad \varepsilon'' = \gamma'',$$

the three first of which are the values of B, B′, B″ (8) taken with a change of signs. In both cases here implied, the values of α, α', &c. will be as in formulæ (A); and we have taken $\theta = t' + t''$, $K = \frac{1}{2} R k \theta$.

The formulæ (E) are regular in the composition of their terms, and but little more complex than (A), from which they chiefly differ in this respect, that the factor k cannot be eliminated, but must correspond to a determinate value of the comet's distance r. Any change in the value of this distance will however be attended only with small additional computation; since the calculated values of the different terms in (E) will, from the form which we have adopted, continue invariable.

When the intervals t', t'' happen to be equal, we shall have $K = R k t'$, and the more simple values

$$\left.\begin{aligned} x_{,} &= X_{,} + \frac{\rho\mu'}{E t'}(E'\alpha' - E''\alpha'') + K\left(\frac{\varepsilon''\alpha'}{E} - \frac{1}{2} + \frac{X_{,}t'}{6R}\right), \\ y_{,} &= Y_{,} + \frac{\rho\mu'}{E t'}(E'\beta' - E''\beta'') + K\left(\frac{\varepsilon''\beta'}{E} + \frac{Y_{,}t'}{6R}\right), \\ z_{,} &= \frac{\rho\mu'}{E t'}(E'\gamma' - E''\gamma'') + K\,\frac{\varepsilon''\gamma'}{E}; \end{aligned}\right\} \quad (E')$$

the quantities E, E′, E″, ε'' being as in (E), and the values of α, α', &c. the same as in the particular forms (A′).

The investigation of the formulæ now given, we reserve for another paper, which will soon appear. At present we shall briefly show how the expressions given by Mr Pontécoulant may be readily deduced from our expressions (19).

If in these we omit the last terms depending on the earth's veloci-

ties $X_{,}$, $Y_{,}$; and put $\mu' = 1$, $\mu'' = 1$, &c. and also $\rho = h\,M\,t'\,t''$, there will result for the component velocities $x_{,}$, $y_{,}$, $z_{,}$, the less approximate values

$$\left.\begin{aligned} x_{,} &= X_{,} + h\,(t'M'\alpha' - t''M''\alpha'' - (t' - t'')\,D\xi_{,}), \\ y_{,} &= Y_{,} + h\,(t'M'\beta' - t''M''\beta'' - (t' - t'')\,D\eta_{,}), \\ z_{,} &= h\,(t'M'\gamma' - t''M''\gamma''); \end{aligned}\right\} \qquad (a)$$

and if to these be added the identical equations

$$\begin{aligned} &M\alpha + M'\alpha' + M''\alpha'' - D\xi_{,} = 0, \\ &M\beta + M'\beta' + M''\beta'' - D\eta_{,} = 0, \\ &M\gamma + M'\gamma' + M''\gamma'' = 0, \end{aligned}$$

respectively multiplied by $\frac{1}{2}\,h\,(t'' - t')$, we shall immediately obtain the following expressions:

$$\left.\begin{aligned} x_{,} &= X_{,} + h\,\{\tfrac{1}{2}\,\theta\,(M'\alpha' - M''\alpha'') - \tfrac{1}{2}\,\theta'\,(M\alpha + D\xi_{,})\} \\ y_{,} &= Y_{,} + h\,\{\tfrac{1}{2}\,\theta\,(M'\beta' - M''\beta'') - \tfrac{1}{2}\,\theta'\,(M\beta + D\eta_{,})\} \\ z_{,} &= h\,\{\tfrac{1}{2}\,\theta\,(M'\gamma' - M''\gamma'') - \tfrac{1}{2}\,\theta'\,M\gamma\}; \end{aligned}\right\} \qquad (b)$$

in which $\theta = t' + t''$, $\theta' = t' - t''$.

These are in fact less simple than the preceding. They become identical with the values $F_{,}$, $G_{,}$, $H_{,}$, found at page 44, vol. II. Théorie Analytique, &c. when we further neglect the arc a in the values of $\xi_{,}$, $\eta_{,}$, multiplied by the factor D. The term $D\xi_{,}$ may be removed by directing the axis of y to the earth when the comet is at C. The preceding values of $x_{,}$, $y_{,}$ will then agree with $P_{,}$, $Q_{,}$, page 45. But even in this simplified state, the computation depends on nine different angles instead of the three involved in formulæ (A).

The preceding values (a) and (b) are not the only forms that can be derived from (19), in virtue of the identical equations which we have just employed. If to (a) we add those equations respectively

multiplied by $h\ (t''-t')$, the terms which depend on D will disappear, and we shall obtain a new set of values:

$$\left.\begin{aligned} x_{,} &= X_{,} + h\,\{t''M'\alpha' - t'M''\alpha'' - (t'-t'')M\alpha\}, \\ y_{,} &= Y_{,} + h\,\{t''M'\beta' - t'M''\beta'' - (t'-t'')M\beta\}, \\ z_{,} &= h\,\{t''M'\gamma' - t'M''\gamma'' - (t'-t'')M\gamma\}; \end{aligned}\right\} \qquad (c)$$

which are perfectly symmetrical, and preferable to (b), both as to form and facility of computation.

We presume that the advantage of our mode of solution is sufficiently tested by the different results we have obtained. We shall therefore proceed to the numerical application of formulæ (A). In the example chosen, the intervals t', t'' are considerably different, and $t'-t''$ is of no small magnitude. We have taken it as before mentioned from the Théorie Analytique du Système du Monde, vol. II., p. 68.

Data of Comet of 1805.

Times of Observation.	Longitudes Observed.	Latitudes Observed.	Intervals.
Nov. 23^d·32241	$l' = 24°41'04''$	$\lambda' = 27°25'35''$	$t' = 7^d$·18854
" 30·51095	$l = 15\ 39\ 40$	$\lambda = 19\ 25\ 28$	$t'' = 4$·78486
Dec. 5·29581	$l'' = 2\ 7\ 11$	$\lambda'' = 3\ 20\ 45$	

These have been corrected for aberration and parallax, and with reference to mean time at Paris.

From the tables of Delambre have been taken

(Long. of ⊙ + 180°) or $L = 68°\ 25'\ 41''$; $\log R = 9{\cdot}9936673$.

To these I add from values before noticed;

$\varpi = 99°\ 34'\ 13''$; $L - \varpi = 31°\ 8'\ 32''$; $\sin\varepsilon =$ ·01685151.

Preliminary Quantities for (A).

$t' = 9{\cdot}0922228$	$p = 7{\cdot}9403288$	$a + l' - L = -42°55'51''$	
$t'' = 8{\cdot}9154514$	$q = 0{\cdot}0062710$	$a + l - L = -51\ 57\ 15$	$s = 9{\cdot}9937738$
$t' - t'' = 8{\cdot}6164587$	$a = +48'46''$	$a + l'' - L = -65\ 29\ 44$	

The numbers in this and the following are sufficiently distinguished from the logarithms by the prefixed signs + and —

From Computation of Formulæ (A).

Values of $X_{\prime}$, $Y_{\prime}$.

$p \cos a - q \sin a$	$p \sin a + q \cos a$
$- {\cdot}0087163 - {\cdot}0143914$	$- {\cdot}0001236 + 1{\cdot}0144421$
$X_{\prime} = - {\cdot}0231077.$	$Y_{\prime} = + 1{\cdot}0143185.$

Values of A, A′, A″, &c.

$\beta'\gamma'' - \beta''\gamma'$	$\beta''\gamma - \beta\gamma''$	$\beta\gamma' - \beta'\gamma$
$-{\cdot}0398195 + {\cdot}4722050$	$-{\cdot}3208804 + {\cdot}0460401$	$-{\cdot}4086707 + {\cdot}2401851$
$A = + {\cdot}4323855.$	$A' = - {\cdot}2748403.$	$A'' = - {\cdot}1684856.$

At'	At''	$At't''$	$A\alpha$
$8{\cdot}7280941$	$8{\cdot}5513227$	$7{\cdot}6435455$	$+ {\cdot}2664756$

$\rho\left(\frac{A'\alpha'}{At''} - \frac{A''\alpha''}{At'}\right)$	$\rho\left(\frac{A'\beta'}{At''} - \frac{A''\beta''}{At'}\right)$	$\rho\left(\frac{A'\gamma'}{At''} - \frac{A''\gamma''}{At'}\right)$
$0{\cdot}7523734;\ 0{\cdot}1162696$	$0{\cdot}7209781;\ 0{\cdot}4574862$	$0{\cdot}6028708;\ 9{\cdot}2653444$
$-5{\cdot}654229 + 1{\cdot}306982$	$+5{\cdot}259907 - 2{\cdot}867386$	$-4{\cdot}007474 + 0{\cdot}184223$
$-\rho(4{\cdot}347247).$	$+\rho(2{\cdot}392521).$	$-\rho(3{\cdot}823251).$

We have supposed in this computation that the factors μ, μ', &c. are each unity. But if we take for greater precision the distance $r = 1$,

$\log \mu'' = 9{\cdot}9990181$; $\log \mu' = 9{\cdot}9977807$; $\log \mu = 9{\cdot}9992627$; $\log \nu = 9{\cdot}9985244$,

we shall find the corresponding values,

$$-\rho\,(4{\cdot}341140). \qquad +\rho\,(2{\cdot}395257). \qquad -\rho\,(3{\cdot}815140).$$

If to the value of $A\alpha = {\cdot}2664756$, we join $A'\alpha' = -{\cdot}2012315$, $A''\alpha'' = -{\cdot}0698817$, obtained from the logarithms used in the preceding computation, there will result $D = -{\cdot}0046376$, and then we shall get

$\rho\left(D\frac{t''-t'}{t't''} + D\frac{X_{,}}{3As}\right)$	$\rho\left(D\frac{Y_{,}}{3As}\right)$
8·6392065; 5·9232877.	7·5657013
+ ·0435719 + ·0000838	$-\rho$ (·003679).
$+\rho$ (·043655).	

In these values we have taken $\mu = 1$, $\mu' = 1$, &c. If we employ the logarithms which correspond to $r = 1$, there will result, instead of the preceding:

$$+\rho\,({\cdot}043581), \qquad -\rho\,({\cdot}003662);$$

by virtue of which and the value before given, we obtain for the comet's velocities:

$$\left.\begin{aligned} x_{,} &= -0{\cdot}023108 - \rho\,(4{\cdot}297559), \\ y_{,} &= +1{\cdot}014319 + \rho\,(2{\cdot}391595), \\ z_{,} &= -\rho\,(3{\cdot}815140); \end{aligned}\right\} \qquad (d)$$

in which ρ is the curtate distance from the centre of the earth.

Had we retained the values found in case of $\mu = 1$, $\mu' = 1$, &c., the coefficients of ρ would have been

$$(4{\cdot}303592), \qquad (2{\cdot}388842), \qquad (3{\cdot}823251);$$

which are not so accurate as those in (d).

The preceding expressions for $x_{,}$, $y_{,}$, $z_{,}$, may be easily changed so as to correspond to any required position of the axis of x, and to the ray drawn from the centre E of the earth to the comet at C. If we sup-

posed this axis to be directed to E, we shall find in terms of ρ before used;

$$\left.\begin{aligned} x_{,} &= -0\cdot008716 - \rho\,(4\cdot263202), \\ y_{,} &= +1\cdot014544 + \rho\,(2\cdot452316), \\ z_{,} &= -\rho\,(3\cdot815140). \end{aligned}\right\} \qquad (e)$$

But if we also change ρ into $\rho \cos \lambda$, so that ρ may denote the ray CE, we shall get

$$\begin{aligned} x_{,} &= -0\cdot008716 - \rho\,(4\cdot020544), \\ y_{,} &= +1\cdot014544 + \rho\,(2\cdot312732), \\ z_{,} &= -\rho\,(3\cdot597985). \end{aligned}$$

The accuracy of these results we believe to be very considerable. Mr Pontécoulant's values in terms of ρ are the following:

$x_{,} = -0\cdot008686 - \rho\,(4\cdot026273)$, $y_{,} = +1\cdot014545 + \rho\,(2\cdot314020)$;

$z_{,} = -\rho\,(3\cdot605632)$. (See Théorie Analytique, &c., vol. II., p. 70.)

With the desire of making a comparison between the principal terms of formulæ (E), and the values (d) above given, I have subjected them to computation. By taking the first expressions for E, E′, E″ and ε'', the coefficients of the curtate distance ρ were as here given:

$$\begin{aligned} x_{,} &= X_{,} - \rho\,(4\cdot050920) - \&c. \\ y_{,} &= Y_{,} + \rho\,(2\cdot047392) + \&c. \\ z_{,} &= -\rho\,(3\cdot646140) - \&c. \end{aligned}$$

so that the terms affected by K differ little from $-\rho\,(\cdot245653)$, $+\rho\,(\cdot345974)$, $-\rho\,(\cdot168997)$, and will in the destined use of (E) not exceed the order of the intervals t', t''.

To the data of the example now considered, I have been induced to apply the method of La Place; and with surprise I found results

which it may be proper here to insert. By means of the two expressions

$$dl = \frac{(l''-l)\,t'^2 + (l-l')\,t''^2}{t'\,t''\,(t'+t'')}, \quad d^2 l = \frac{(l''-l)\,t' - (l-l')\,t''}{t'\,t''\,(t'+t'')};$$

which are deducible from the values of l', l'', expressed by Maclaurin's theorem; I obtained in parts of radius, $d\,l = -2{\cdot}232849$; $d^2\,l = -15{\cdot}51507$. The similar values which depend on the geocentric latitudes were $d\,\lambda = -2{\cdot}498243$, $d^2\,\lambda = -22{\cdot}139036$; and adopting in this method the formulæ:

$$d\,\rho = \frac{m}{n}\,\rho, \quad m = d^2 l + \tan(\mathrm{L}-l)\left(d l^2 + \frac{d^2\lambda}{\sin\lambda\cos\lambda} + \frac{2\,d\,\lambda^2}{\cos\lambda^2}\right),$$

$$n = 2\,d\,l + 2\tan(\mathrm{L}-l)\frac{d\,\lambda}{\sin\lambda\cos\lambda};$$

which may be found expressed differently in the Mécanique Celeste and Theorie Analytique, &c. I obtained $\frac{m}{n}$ or $i = -3{\cdot}278562$. The three equations

$$x_{,} = \mathrm{X}_{,} + \rho\,(i\,\alpha - \beta\,d\,l),$$

$$y_{,} = \mathrm{Y}_{,} + \rho\,(i\,\beta + \alpha\,d\,l),$$

$$z_{,} = \rho\left(i\,\gamma + \frac{d\lambda}{\cos\lambda}\right),$$

resulting from the differentiation of (1), and in which $\alpha = \cos(l-\mathrm{L})$, $\beta = \sin(l-\mathrm{L})$, $\gamma = \tan\lambda$, then gave me the following values:

$$x_{,} = \mathrm{X}_{,} - \rho\,(3{\cdot}761473),$$

$$y_{,} = \mathrm{Y}_{,} + \rho\,(1{\cdot}261091),$$

$$z_{,} = -\rho\,(3{\cdot}965141);$$

which are exceedingly different from (*e*), with regard to the coefficients of ρ in the values of $x_{,}$ and $y_{,}$.

In addition to these results, I would join the remark, that the computed values of $x_{,}$, $y_{,}$, $z_{,}$, expressed by the preceding differential formulæ, cannot be made to agree with (*e*), so long as are employed the above numerical quantities found for dl, $d\lambda$; and that from this and other instances, I consider as defective the manner in which the first and second differentials of the geocentric longitudes and latitudes are determined in the method here noticed; and which, I think, should not be used for determining the component velocities of a comet, or the position of its orbit.

We might now particularize further the expressions (*d*) by means of (25) and (26), and the consequent values of the distances r and ρ. But our principal aim in the present paper has, we imagine, been sufficiently attained. We intend again to resume the subject on an early occasion.